Isoperimetric Problems of the Calculus of Variations on Time Scales

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Abstract. We prove a necessary optimality condition for isoperimetric problems on time scales in the space of delta-differentiable functions with rd-continuous derivatives. The results are then applied to Sturm-Liouville eigenvalue problems on time scales.

1. Introduction

The theory of time scales (see Section 2 for basic definitions and results) is a relatively new area, that unify and generalize difference and differential equations [**8**]. It was initiated by Stefan Hilger in the nineties of the XX century [**12, 13**], and is now subject of strong current research in many different fields in which dynamic processes can be described with discrete or continuous models [**1**].

The study of the calculus of variations on time scales has began in 2004 with the paper [**6**] of Bohner, where the necessary optimality conditions of Euler-Lagrange and Legendre, as well as a sufficient Jacobi-type condition, are proved for the basic problem of the calculus of variations with fixed endpoints. Since the pioneer paper [**6**], the following classical results of the calculus of variations on continuous-time ($\mathbb{T} = \mathbb{R}$) and discrete-time ($\mathbb{T} = \mathbb{Z}$) have been unified and generalized to a time scale $\mathbb{T}$: the Noether's theorem [**5**]; the Euler-Lagrange equations for problems of the calculus of variations with double integrals [**7**] and for problems with higher-order derivatives [**10**]; transversality conditions [**14**]. The more general theory of the calculus of variations on time scales seems to be useful in applications to Economics [**4**]. Much remains to be done [**11**], and here we give a step further. Our main aim is to obtain a necessary optimality condition for isoperimetric problems on time scales. Corollaries include the classical case ($\mathbb{T} = \mathbb{R}$), which is extensively studied in the literature (see, e.g., [**15**]); and discrete-time versions [**3**].

The plan of the paper is as follows. Section 2 gives a short introduction to time scales, providing the definitions and results needed in the sequel. In Section 3 we prove a necessary optimality condition for the isoperimetric problem on time scales

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(Theorem 3.4); then, we establish a connection (Theorem 3.7) with the previously studied Sturm-Liouville eigenvalue problems on time scales [**2**].

2. The calculus on time scales and preliminaries

We begin by recalling the main definitions and properties of time scales (cf. [**1, 8, 12, 13**] and references therein).

A nonempty closed subset of $\mathbb{R}$ is called a *Time Scale* and is denoted by $\mathbb{T}$. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, for all $t \in \mathbb{T}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, for all $t \in \mathbb{T}$, with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m). A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* and *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively. Throughout the text we let $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$ with $a, b \in \mathbb{T}$. We define $\mathbb{T}^\kappa = \mathbb{T} \backslash (\rho(b), b]$ and $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$, for all $t \in \mathbb{T}$. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^\kappa$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t. For delta differentiable f and g, the next formulas hold:

$$\begin{aligned} f^\sigma(t) &= f(t) + \mu(t) f^\Delta(t)\,, \\ (fg)^\Delta(t) &= f^\Delta(t) g^\sigma(t) + f(t) g^\Delta(t) \\ &= f^\Delta(t) g(t) + f^\sigma(t) g^\Delta(t), \end{aligned} \tag{2.1}$$

where we abbreviate $f \circ \sigma$ by f^σ. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous in right-dense points and if its left-sided limit exists in left-dense points. We denote the set of all rd-continuous functions by C_{rd} or $\mathrm{C}_{\mathrm{rd}}[\mathbb{T}]$ and the set of all delta differentiable functions with rd-continuous derivative by $\mathrm{C}^1_{\mathrm{rd}}$ or $\mathrm{C}^1_{\mathrm{rd}}[\mathbb{T}]$. It is useful to provide an example to the reader with the concepts introduced so far. Consider $\mathbb{T} = \bigcup_{k=0}^\infty [2k, 2k + 1]$. For this time scale,

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^\infty [2k, 2k + 1); \\ 1 & \text{if } t \in \bigcup_{k=0}^\infty \{2k + 1\}. \end{cases}$$

Let us consider $t \in [0, 1] \cap \mathbb{T}$. Then, we have (see [**8**, Theorem 1.16])

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},\ t \in [0, 1)\,,$$

provided this limit exists, and

$$f^\Delta(1) = \frac{f(2) - f(1)}{2 - 1},$$

provided f is continuous at $t = 1$. Let

$$f(t) = \begin{cases} t & \text{if } t \in [0, 1); \\ 2 & \text{if } t = 1. \end{cases}$$

We observe that at $t = 1$ f is rd-continuous (since $\lim_{t \to 1} f(t) = 1$) but not continuous (since $f(1) = 2$).

It is known that rd-continuous functions possess an *antiderivative*, i.e., there exists a function F with $F^{\Delta} = f$, and in this case an *integral* is defined by $\int_a^b f(t)\Delta t = F(b) - F(a)$. It satisfies

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)\,. \tag{2.2}$$

Lemma 2.1 gives the integration by parts formulas of the delta integral:

LEMMA 2.1 ([**8**]). *If $a, b \in \mathbb{T}$ and $f, g \in C^1_{rd}$, then*

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^{\Delta}(t)g(t)\Delta t, \tag{2.3}$$

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t. \tag{2.4}$$

The following time scale DuBois-Reymond lemma will be useful for our purposes:

LEMMA 2.2 ([**6**]). *Let $g \in C_{rd}$, $g : [a, b]^{\kappa} \to \mathbb{R}^n$. Then,*

$$\int_a^b g^T(t)\eta^{\Delta}(t)\Delta t = 0, \textit{for all } \eta \in C^1_{rd} \textit{ with } \eta(a) = \eta(b) = 0$$

holds if and only if

$$g(t) = c, \textit{on } [a, b]^{\kappa} \textit{ for some } c \in \mathbb{R}^n.$$

Finally, we prove a simple but useful technical lemma.

LEMMA 2.3. *Suppose that a continuous function $f : \mathbb{T} \to \mathbb{R}$ is such that $f^{\sigma}(t) = 0$ for all $t \in \mathbb{T}^{\kappa}$. Then, $f(t) = 0$ for all $t \in \mathbb{T}\backslash\{a\}$ if a is right-scattered.*

PROOF. First note that, since $f^{\sigma}(t) = 0$, then $f^{\sigma}(t)$ is delta differentiable, hence continuous for all $t \in \mathbb{T}^{\kappa}$. Now, if t is right-dense, the result is obvious. Suppose that t is right-scattered. We will analyze two cases: (i) if t is left-scattered, then $t \neq a$ and by hypothesis $0 = f^{\sigma}(\rho(t)) = f(t)$; (ii) if t is left-dense, then, by the continuity of f^{σ} and f at t, we can write

$$\forall\varepsilon > 0\, \exists\delta_1 > 0 : \forall s_1 \in (t - \delta_1, t], \text{ we have } |f^{\sigma}(s_1) - f^{\sigma}(t)| < \varepsilon\,, \tag{2.5}$$

$$\forall\varepsilon > 0\, \exists\delta_2 > 0 : \forall s_2 \in (t - \delta_2, t], \text{ we have } |f(s_2) - f(t)| < \varepsilon\,, \tag{2.6}$$

respectively. Let $\delta = \min\{\delta_1, \delta_2\}$ and take $s_1 \in (t - \delta, t)$. As $\sigma(s_1) \in (t - \delta, t)$, take $s_2 = \sigma(s_1)$. By (2.5) and (2.6), we have:

$$|-f^{\sigma}(t)+f(t)| = |f^{\sigma}(s_1)-f^{\sigma}(t)+f(t)-f(s_2)| \leq |f^{\sigma}(s_1)-f^{\sigma}(t)|+|f(s_2)-f(t)| < 2\varepsilon.$$

Since ε is arbitrary, $|-f^{\sigma}(t) + f(t)| = 0$, which is equivalent to $f(t) = f^{\sigma}(t)$. $\square$

3. Main results

We start in §3.1 by defining the isoperimetric problem on time scales and proving a correspondent first-order necessary optimality condition (Theorem 3.4). Then, in §3.2, we show that certain eigenvalue problems can be recast as an isoperimetric problem (Theorem 3.7).

3.1. Isoperimetric problems. Let $J : \mathrm{C}^1_{\mathrm{rd}} \to \mathbb{R}$ be a functional defined on the function space $(\mathrm{C}^1_{\mathrm{rd}}, \|\cdot\|)$ and let $S \subseteq \mathrm{C}^1_{\mathrm{rd}}$.

DEFINITION 3.1. The functional J is said to have a *local minimum* in S at $y_* \in S$ if there exists a $\delta > 0$ such that $J(y_*) \leq J(y)$ for all $y \in S$ satisfying $\|y - y_*\| < \delta$.

Now, let $J : \mathrm{C}^1_{\mathrm{rd}} \to \mathbb{R}$ be a functional of the form

$$J(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t, \tag{3.1}$$

where $L(t,x,v) : [a,b]^\kappa \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has continuous partial derivatives $L_x(t,x,v)$ and $L_v(t,x,v)$, respectively with respect to the second and third variables, for all $t \in [a,b]^\kappa$, and is such that $L(t, y^\sigma(t), y^\Delta(t))$, $L_x(t, y^\sigma(t), y^\Delta(t))$ and $L_v(t, y^\sigma(t), y^\Delta(t))$ are rd-continuous in t for all $y \in \mathrm{C}^1_{\mathrm{rd}}$. The *isoperimetric problem* consists of finding functions y satisfying given boundary conditions

$$y(a) = y_a, \; y(b) = y_b, \tag{3.2}$$

and a constraint of the form

$$I(y) = \int_a^b g(t, y^\sigma(t), y^\Delta(t))\Delta t = l, \tag{3.3}$$

where $g(t,x,v) : [a,b]^\kappa \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has continuous partial derivatives with respect to the second and third variables for all $t \in [a,b]^\kappa$, $g(t, y^\sigma(t), y^\Delta(t))$, $g_x(t, y^\sigma(t), y^\Delta(t))$ and $g_v(t, y^\sigma(t), y^\Delta(t))$ are rd-continuous in t for all $y \in \mathrm{C}^1_{\mathrm{rd}}$, and l is a specified real constant, that takes (3.1) to a minimum.

DEFINITION 3.2. We say that a function $y \in \mathrm{C}^1_{\mathrm{rd}}$ is *admissible* for the isoperimetric problem if it satisfies (3.2) and (3.3).

DEFINITION 3.3. An admissible function y_* is said to be an *extremal* for I if it satisfies the following equation (cf. [**6**]):

$$g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_x(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau = c,$$

for all $t \in [a,b]^\kappa$ and some constant c.

THEOREM 3.4. *Suppose that J has a local minimum at $y_* \in C^1_{rd}$ subject to the boundary conditions (3.2) and the isoperimetric constraint (3.3), and that y_* is not an extremal for the functional I. Then, there exists a Lagrange multiplier constant λ such that y_* satisfies the following equation:*

$$F_v^\Delta(t, y_*^\sigma(t), y_*^\Delta(t)) - F_x(t, y_*^\sigma(t), y_*^\Delta(t)) = 0, \; \textit{for all } t \in [a,b]^{\kappa^2}, \tag{3.4}$$

where $F = L - \lambda g$ and F_v^Δ denotes the delta derivative of a composition.

PROOF. Let y_* be a local minimum for J and consider neighboring functions of the form

$$\hat{y} = y_* + \varepsilon_1\eta_1 + \varepsilon_2\eta_2, \tag{3.5}$$

where for each $i \in \{1,2\}$, ε_i is a sufficiently small parameter (ε_1 and ε_2 must be such that $\|\hat{y} - y^*\| < \delta$, for some $\delta > 0$ – see Definition 3.1), $\eta_i(x) \in \mathrm{C}^1_{\mathrm{rd}}$ and $\eta_i(a) = \eta_i(b) = 0$. Here, η_1 is an arbitrary fixed function and η_2 is a fixed function that we will choose later.

First we show that (3.5) has a subset of admissible functions for the isoperimetric problem. Consider the quantity

$$I(\hat{y}) = \int_a^b g(t, y_*^\sigma(t) + \varepsilon_1 \eta_1^\sigma(t) + \varepsilon_2 \eta_2^\sigma(t), y_*^\Delta(t) + \varepsilon_1 \eta_1^\Delta(t) + \varepsilon_2 \eta_2^\Delta(t))\Delta t.$$

Then we can regard $I(\hat{y})$ as a function of ε_1 and ε_2, say $I(\hat{y}) = \hat{Q}(\varepsilon_1, \varepsilon_2)$. Since y_* is a local minimum for J subject to the boundary conditions and the isoperimetric constraint, putting $Q(\varepsilon_1, \varepsilon_2) = \hat{Q}(\varepsilon_1, \varepsilon_2) - l$ we have that

$$Q(0,0) = 0. \tag{3.6}$$

By the conditions imposed on g, we have

$$\begin{aligned} \frac{\partial Q}{\partial \varepsilon_2}(0,0) &= \int_a^b \left[g_x(t, y_*^\sigma(t), y_*^\Delta(t))\eta_2^\sigma(t) + g_v(t, y_*^\sigma(t), y_*^\Delta(t))\eta_2^\Delta(t) \right] \Delta t \\ &= \int_a^b \left[g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_x(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau \right] \eta_2^\Delta(t)\Delta t, \end{aligned} \tag{3.7}$$

where (3.7) follows from (2.3) and the fact that $\eta_2(a) = \eta_2(b) = 0$. Now, the function

$$E(t) = g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_x(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau$$

is rd-continuous on $[a,b]^\kappa$. Hence, we can apply Lemma 2.2 to show that there exists a function $\eta_2 \in \mathrm{C}^1_{\mathrm{rd}}$ such that

$$\int_a^b \left[g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_x(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau \right] \eta_2^\Delta(t)\Delta t \neq 0,$$

provided y_* is not an extremal for I, which is indeed the case. We have just proved that

$$\frac{\partial Q}{\partial \varepsilon_2}(0,0) \neq 0. \tag{3.8}$$

Using (3.6) and (3.8), the implicit function theorem asserts that there exist neighborhoods N_1 and N_2 of 0, $N_1 \subseteq \{\varepsilon_1 \text{ from } (3.5)\} \cap \mathbb{R}$ and $N_2 \subseteq \{\varepsilon_2 \text{ from } (3.5)\} \cap \mathbb{R}$, and a function $\varepsilon_2 : N_1 \to \mathbb{R}$ such that for all $\varepsilon_1 \in N_1$ we have

$$Q(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0,$$

which is equivalent to $\hat{Q}(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = l$. Now we derive the necessary condition (3.4). Consider the quantity $J(\hat{y}) = K(\varepsilon_1, \varepsilon_2)$. By hypothesis, K is minimum at $(0,0)$ subject to the constraint $Q(0,0) = 0$, and we have proved that $\nabla Q(0,0) \neq \mathbf{0}$. We can appeal to the Lagrange multiplier rule (see, e.g., [**15**, Theorem 4.1.1]) to assert that there exists a number λ such that

$$\nabla(K(0,0) - \lambda Q(0,0)) = \mathbf{0}. \tag{3.9}$$

Having in mind that $\eta_1(a) = \eta_1(b) = 0$, we can write:

$$\begin{aligned} \frac{\partial K}{\partial \varepsilon_1}(0,0) &= \int_a^b \left[L_x(t, y_*^\sigma(t), y_*^\Delta(t))\eta_1^\sigma(t) + L_v(t, y_*^\sigma(t), y_*^\Delta(t))\eta_1^\Delta(t) \right] \Delta t \\ &= \int_a^b \left[L_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t L_x(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau \right] \eta_1^\Delta(t)\Delta t. \end{aligned} \tag{3.10}$$

Similarly, we have that

$$\frac{\partial Q}{\partial \varepsilon_1}(0,0) = \int_a^b \left[g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_x(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau \right] \eta_1^\Delta(t)\Delta t. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11), we obtain

$$\int_a^b \left\{ L_v(\cdot) - \int_a^t L_x(\cdot\cdot)\Delta\tau - \lambda \left(g_v(\cdot) - \int_a^t g_x(\cdot\cdot)\Delta\tau \right) \right\} \eta_1^\Delta(t)\Delta t = 0,$$

where $(\cdot) = (t, y_*^\sigma(t), y_*^\Delta(t))$ and $(\cdot\cdot) = (\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))$. Since η_1 is arbitrary, Lemma 2.2 implies that there exists a constant d such that

$$L_v(\cdot) - \lambda g_v(\cdot) - \left(\int_a^t [L_x(\cdot\cdot) - \lambda g_x(\cdot\cdot)]\Delta\tau \right) = d, \ \ t \in [a,b]^\kappa,$$

or

$$F_v(\cdot) - \int_a^t F_x(\cdot\cdot)\Delta\tau = d, \tag{3.12}$$

with $F = L - \lambda g$. Since the integral and the constant in (3.12) are delta differentiable, we obtain the desired necessary optimality condition (3.4). $\square$

Remark 3.5. Theorem 3.4 remains valid when y_* is assumed to be a local maximizer of the isoperimetric problem (3.1)-(3.3).

Example 3.6. Suppose that we want to find functions defined on $[-a,a] \cap \mathbb{T}$ that take

$$J(y) = \int_{-a}^a y^\sigma(t)\Delta t$$

to its largest value (see Remark 3.5) and that satisfy the conditions

$$y(-a) = y(a) = 0, \quad I(y) = \int_{-a}^a \sqrt{1 + (y^\Delta(t))^2}\Delta t = l > 2a.$$

Note that if y is an extremal for I, then y is a line segment [**6**], and therefore $y(t) = 0$ for all $t \in [-a,a]$. This implies that $I(y) = 2a > 2a$, which is a contradiction. Hence, I has no extremals satisfying the boundary conditions and the isoperimetric constraint. Using Theorem 3.4, let

$$F = L - \lambda g = y^\sigma - \lambda\sqrt{1 + (y^\Delta)^2}\,.$$

Because

$$F_x = 1, \quad F_v = \lambda \frac{y^\Delta}{\sqrt{1 + (y^\Delta)^2}},$$

a necessary optimality condition is given by the following delta-differential equation:

$$\lambda \left(\frac{y^\Delta}{\sqrt{1 + (y^\Delta)^2}} \right)^\Delta - 1 = 0, \quad t \in [-a,a]^{\kappa^2}.$$

The reader interested in the study of delta-differential equations on time scales is referred to [**9**] and references therein.

If we restrict ourselves to times scales $\mathbb{T}$ with $\sigma(t) = a_1 t + a_0$ for some $a_1 \in \mathbb{R}^+$ and $a_0 \in \mathbb{R}$ ($a_0 = 0$ and $a_1 = 1$ for $\mathbb{T} = \mathbb{R}$; $a_0 = a_1 = 1$ for $\mathbb{T} = \mathbb{Z}$), it follows from the results in [**10**] that the same proof of Theorem 3.4 can be used, *mutatis mutandis*, to obtain a necessary optimality condition for the higher-order isoperimetric problem (i.e., when L and g contain higher order delta derivatives). In this case, the necessary optimality condition (3.4) is generalized to

$$\sum_{i=0}^{r}(-1)^i \left(\frac{1}{a_1}\right)^{\frac{(i-1)i}{2}} F_{u_i}^{\Delta^i}\left(t, y_*^{\sigma^r}(t), y_*^{\sigma^{r-1}\Delta}(t), \ldots, y_*^{\sigma\Delta^{r-1}}(t), y_*^{\Delta^r}(t)\right) = 0\,,$$

where $F = L - \lambda g$, and functions $(t, u_0, u_1, \ldots, u_r) \to L(t, u_0, u_1, \ldots, u_r)$ and $(t, u_0, u_1, \ldots, u_r) \to g(t, u_0, u_1, \ldots, u_r)$ are assumed to have (standard) partial derivatives with respect to $u_0, \ldots, u_r$, $r \geq 1$, and partial delta derivative with respect to t of order $r + 1$.

3.2. Sturm-Liouville eigenvalue problems. Eigenvalue problems on time scales have been studied in [**2**]. Consider the following Sturm-Liouville eigenvalue problem: find nontrivial solutions to the delta-differential equation

$$y^{\Delta^2}(t) + q(t)y^\sigma(t) + \lambda y^\sigma(t) = 0,\ t \in [a, b]^{\kappa^2}, \tag{3.13}$$

for the unknown $y : [a, b] \to \mathbb{R}$ subject to the boundary conditions

$$y(a) = y(b) = 0. \tag{3.14}$$

Here $q : [a, b] \to \mathbb{R}$ is a continuous function and $y^{\Delta^2} = (y^\Delta)^\Delta$.

Generically, the only solution to equation (3.13) that satisfies the boundary conditions (3.14) is the trivial solution, $y(t) = 0$ for all $t \in [a, b]$. There are, however, certain values of λ that lead to nontrivial solutions. These are called *eigenvalues* and the corresponding nontrivial solutions are called *eigenfunctions*. These eigenvalues may be arranged as $-\infty < \lambda_1 < \lambda_2 < \ldots$ (see Theorem 1 of [**2**]) and λ_1 is called the *first eigenvalue*.

Consider the functional defined by

$$J(y) = \int_a^b ((y^\Delta)^2(t) - q(t)(y^\sigma)^2(t))\Delta t, \tag{3.15}$$

and suppose that $y_* \in \mathrm{C}^2_{\mathrm{rd}}$ (functions that are twice delta differentiable with rd-continuous second delta derivative) is a local minimum for J subject to the boundary conditions (3.14) and the isoperimetric constraint

$$I(y) = \int_a^b (y^\sigma)^2(t)\Delta t = 1. \tag{3.16}$$

If y_* is an extremal for I, then we would have $-2y^\sigma(t) = 0,\ t \in [a, b]^\kappa$. Noting that $y(a) = 0$, using Lemma 2.3 we would conclude that $y(t) = 0$ for all $t \in [a, b]$. No extremals for I can therefore satisfy the isoperimetric condition (3.16). Hence, by Theorem 3.4 there is a constant λ such that y_* satisfies

$$F^\Delta_{y^\Delta}(t, y_*^\sigma(t), y_*^\Delta(t)) - F_{y^\sigma}(t, y_*^\sigma(t), y_*^\Delta(t)) = 0, \tag{3.17}$$

with $F = (y^\Delta)^2 - q(y^\sigma)^2 - \lambda(y^\sigma)^2$. It is easily seen that (3.17) is equivalent to (3.13). The isoperimetric problem thus corresponds to the Sturm-Liouville problem augmented by the normalizing condition (3.16), which simply scales the eigenfunctions. Here, the Lagrange multiplier plays the role of the eigenvalue.

Theorem 3.7. *Let λ_1 be the first eigenvalue for the Sturm-Liouville problem (3.13) with boundary conditions (3.14), and let y_1 be the corresponding eigenfunction normalized to satisfy the isoperimetric constraint (3.16). Then, among functions in C^2_{rd} that satisfy the boundary conditions (3.14) and the isoperimetric condition (3.16), the functional J defined by (3.15) has a minimum at y_1. Moreover, $J(y_1) = \lambda_1$.*

Proof. Suppose that J has a minimum at y satisfying conditions (3.14) and (3.16). Then y satisfies equation (3.13) and multiplying this equation by y^σ and delta integrating from a to b, we obtain

$$\int_a^b y^\sigma(t) y^{\Delta^2}(t)\Delta t + \int_a^b q(t)(y^\sigma)^2(t)\Delta t + \lambda \int_a^b (y^\sigma)^2(t)\Delta t = 0. \tag{3.18}$$

Since $y(a) = y(b) = 0$,

$$\int_a^b y^\sigma(t) y^{\Delta^2}(t)\Delta t = \left[y(t)y^\Delta(t)\right]_{t=a}^{t=b} - \int_a^b (y^\Delta)^2\Delta t = -\int_a^b (y^\Delta)^2\Delta t,$$

and by (3.16), (3.18) reduces to

$$\int_a^b [(y^\Delta)^2 - q(t)(y^\sigma)^2(t)]\Delta t = \lambda,$$

that is, $J(y) = \lambda$. Due to the isoperimetric condition, y must be a nontrivial solution to (3.13) and therefore λ must be an eigenvalue. Since there exists a least element within the eigenvalues, λ_1, and a corresponding eigenfunction y_1 normalized to meet the isoperimetric condition, the minimum value for J is λ_1 and $J(y_1) = \lambda_1$. □

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